CONSTRUCTING THE PD TERM STRUCTURE

Why the Exponentiation Approach fails and how the PD term structure can be constructed consistently

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ABSTRACT

With the ongoing implementation of the IFRS 9 impairment framework increasing attention is paid to a consistent determination of expected credit lifetime losses. A clear distinction between point-in-time (PIT) and through-the-cycle (TTC) credit default probabilities is well known to be crucial for the construction of suitable PD term structures. However, due to typically insufficient data availability and quality the most widely spread approach to the PD term structure construction, i.e. the **Exponentiation Approach**, still relies on the exponentiation of observed one-year migration matrices irrespective of whether these are derived in a PIT- or TTC-context. In this paper the authors discuss the source of the conceptual inadequacy of this frequently used approach. Addressing these shortcomings, a new and conceptually consistent approach to deriving the genuine PD term structure is introduced and the fixation of its underlying model parameters is benchmarked on synthetic migration time series. This new method, which will be called **Consistent Approach,** allows in particular to estimate the dependence of the genuine PD term structure on the macroeconomic starting condition, rendering it applicable to both regulatory and internal stress testing. Finally, both the Exponentiation Approach and the Consistent Approach outlined in this paper are applied to actual migration data provided by GCD, the Global Credit Data consortium[1], allowing to observe the quantitative impact on PD term structure curves and thus on lifetime expected loss estimates. A profound impact on the term structure is found when comparing the market-standard Exponentiation Approach and the newly developed Consistent Approach.

INTRODUCTION

Methods for modelling the term structure of default risk have been under investigation for a significant period of time.[2] While most of the early model approaches were based on the homogeneous Markov

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[2] R.A. Jarrow, D. Lando, S.M. Turnball: A Markov Model for the Term Structure of Credit Risk Spreads. The Review of Financial Studies. Vol. 10 (2), 1997

assumption[3] the validity of the Markov as well as the homogeneity assumption have soon been put into question by various authors[4,5,6] and advanced models relieving the homogeneity assumption and employing time-inhomogeneous Markov approaches instead have been devised on theoretical grounds.[7],[8] Nonetheless the typical Exponentiation Approach multiplying observed rating migration matrices in accordance with the homogeneity assumption still is the most widely spread market best practice approach in credit risk management today. This is mainly for the reason that only with the introduction of the IFRS 9 accounting standard[9] for impairments − which demands the calculation of expected credit lifetime losses (ECL) given certain trigger criteria − the topic of constructing adequate and consistent PD term structure curves has turned into one of the focus topics of both financial institutions and regulatory authorities.

The IFRS 9 standard prescribes the major properties of such kind of calculation:[10] it has to provide an unbiased probabilistic estimate under a set of scenarios considering historic, current and future economic conditions. With probability of default being one of the major factors in determining ECL, these requirements result in the necessity to estimate unbiased PD term structures over the lifetime of a product, and additionally being able to assess and incorporate the impact of macroeconomic conditions. The latter requirement introduces a link to stress testing, as in future stress tests[11,12] the lifetime expected loss for credit with significant deterioration in credit quality has to be estimated under the stress scenarios as well. These methodological requirements extend well beyond the 1-year PD standard applied for estimating minimum capital requirements.

Some of the aspects required by the new IFRS 9 standard go beyond the achievements of the aforementioned advanced term structure models. In particular, the requirement to compute economy dependent PD term structures, i.e. the temporal evolution of the PIT-PD given the macroeconomic start condition, even if the underlying rating system in operation is not a (pure) PIT rating system, imposes severe intricacies on any term structure methodology. In fact, one finds that the nature of the underlying rating system with respect to its degree of PITness, i.e. the degree to which the rating system is rather PIT than TTC, plays a major role in the derivation of the PD term structure and thus has to be incorporated within the modeling approach.

[3] Under this assumption, the probability of migrating from rating state A into state B (1) is not dependent on how the system happened to evolve into state A (Markov assumption) and (2) additionally does not change over time (homogeneity).
[4] C. Bluhm, L. Overbeck: Calibration of PD Term Structures: To be Markov or not to be. RISK. Vol. 20, No. 11, 2007
[5] D. Landoa, T.M. Skødeberg: Analyzing rating transitions and rating drift with continuous observations. Journal of Banking & Finance. Vol. 26, No. 2–3, 2002
[6] H. Frydmana, T. Schuermann: Credit rating dynamics and Markov mixture models. Journal of Banking & Finance. Vol. 32, Issue 6, 2008
[7] A. Andersson, P. Vanini: Credit migration risk modeling. Journal of Credit Risk, Vol. 6, No. 1, 2010
[8] H. Thompson, J. Harris: Extracting systematic factors in a continuous-time credit migration model. Journal of Credit Risk. Vol. 6, No. 1, 2010
[9] International Accounting Standards Board: IFRS 9 Financial Instruments, 2014, ISBN 978-1-909704-48-0
[10] v.s. section 5.5.17
[11] EBA press release 2016/12/21: https://www.eba.europa.eu/-/eba-to-run-its-next-eu-wide-stress-test-in-2018
[12] EBA draft methodological note: 2018 EU-Wide Stress Test, http://www.eba.europa.eu/documents/10180/1869811/2018+EU-wide+stess+test-Draft+Methodological+Note.pdf, 2017

However, for the sake of clarity it should be stressed that - aside from the emphasis that has been put here on the IFRS 9 requirements - the need for a consistent construction of the PD term structure arises also in the context of many other critical applications such as, for instance, product pricing and credit limit steering.

In this manuscript we propose a PD term structure modeling approach putting particular emphasis on respecting the PITness degree of the underlying rating system. It moreover aims at the clear methodological distinction between systematically driven and idiosyncratic credit migrations allowing thus to explicitly respect the fundamentally different temporal evolution behaviors of these two sources of migrations. The construction of such a model approach thus hinges at least on the following three interlinked questions:

- To which extent is a given rating system point-in-time or through-the-cycle?
- How sensitive is the portfolio of loans to (systematic) macroeconomic changes?
- How can one estimate the idiosyncratic migration rates for credit rating classes?

This manuscript is organized as follows: as a first step the exponentiation approach, i.e. the typical market approach to constructing the PD term structure, is introduced and its limitations are pointed out (section 'Limitations of the Exponentiation Approach'). Secondly, the basic methodology of the here proposed term structure model is discussed (section 'Consistent Approach to the Genuine Term Structure'). A fixation approach of its free model parameters is then introduced on the basis of a likelihood fitting procedure (section 'Fixation of the free Parameters'). Next, the practicability of the proposed term structure model is explored through its application to real world credit default and migration data provided by GCD, the global credit data consortium (section 'A Real World Example'). Finally, a short summary of the main findings and a brief outlook is given (section '**Error! Reference source not found.**').

LIMITATIONS OF THE EXPONENTIATION APPROACH

A reliable estimation of lifetime credit losses requires a sound understanding and modeling of the expected temporal development of the obligor's credit risk as observed through credit rating migrations. The PD term structure $\pi_k(t)$ reflects this relation between time and credit risk. It is defined as the probability of a performing, i.e. non-defaulted, obligor at time t to default in the time interval $[t; t + \Delta t]$ given that the obligor used to be in rating class k at current time t_0 according to

$$\boldsymbol{\pi_k(t) = P(default\ in\ [t; t + \Delta t]\ |\ RC(t_0) = k\ \wedge\ alive\ at\ t)}, \tag{1}$$

where $P(A|B)$ denotes the conditional probability of event A given event B, $RC(t_0)$ is the obligor's rating class at time t_0, $k \in \{1, \dots, N_r\}$, N_r is the number of performing rating classes, the time span Δt is the risk horizon assumed here to be one year, and the second condition effects that only obligors surviving until time t are considered. For clarification it is stated that no default cures will be considered in this paper, meaning that a defaulted obligor is considered to stay defaulted irrevocably.

The formula of the term structure in Eq. (1) thus corresponds to the standard definition of the conditional PD (or forward PD) well known from the general literature in the sense that it is conditional on the survival of the obligor up to the beginning t of the considered time period. However, in the context of this paper a different notation will be used. The term structure specified in Eq. (1) will in fact be referred to as unconditional term structure in the sense that it does not depend on the macroeconomic start condition at t_0.

In contrast to that, the conditional term structure $\pi_k(t, x_0)$ does depend on the macroeconomic condition x_0 at t_0 according to

$$\boldsymbol{\pi_k(t, x_0) = P(default\ in\ [t; t + \Delta t]\ |\ RC(t_0, X) = k\ \wedge\ alive\ at\ t\ \wedge\ X(t_0) = x_0)}, \quad (2)$$

where the random variables $X(t)$ describe the economic state at time t and the rating class $RC(t_0, X)$ can in general also depend on the macroeconomic state as will be discussed in the following. It is this extended definition in Eq. (2) that actually represents the genuine term structure in the sense that it gives an unbiased estimator of the default rate at time t incorporating all available information about the current economic situation at t_0.

However, for the purpose of demonstrating the shortcomings of the Exponentiation Approach the discussion will first be focused here on the unconditional term structure in Eq. (1), since it allows to identify the non-trivial flaws of the Exponentiation Approach in a straight-forward consideration, i.e. the flaws beyond the fact that it cannot model the dependence on the economic start condition.

In principle, the PD term structure could be measured directly following its definition above once a sufficiently extensive set of historic migration time series is available. However, in practice this is typically not the case, rendering the direct measurement approach infeasible due to an often insufficient data recording and a rapidly reducing number of observations at longer maturities especially for the higher risk rating classes.

In practice, the PD term structure is therefore typically measured indirectly. The most widely spread approach, which will be referred to as the Exponentiation Approach in the following, is based on the fundamental idea that the rating migration over N_t time periods could be described as the consecutive application of the one-period migration process repeated for N_t iterations. It is further assumed in this approach that the one-period migration process does not depend on time or any preceding migration events and can thus be described by the exponentiation of the time invariant migration matrix $\bar{M}$, which is typically measured by averaging the historically observed migration matrices according to

$$\boldsymbol{\bar{M} = \frac{1}{N_t} \sum_t M^{obs}(t)}, \quad (3)$$

where $M^{obs}(t)$ denotes the historically observed migration rate matrix for the period $[t; t + \Delta t]$ reflecting the observed relative rates of migrations between two given rating classes including the default state during the considered time period. The idea behind this approach is that the migration rate matrix $M^{obs}(t)$ serves as an estimator of the migration probability matrix $M(t)$ which is defined as

$$M(t) = \begin{pmatrix} m_{1,1}(t) & \dots & m_{1,N_r}(t) & PD_1(t) \\ \vdots & \ddots & \vdots & \vdots \\ m_{N_r,1}(t) & \dots & m_{N_r,N_r}(t) & PD_{N_r}(t) \\ 0 & \dots & 0 & 1 \end{pmatrix} \quad (4)$$

with

$$m_{k,l}(t) = P(RC(t+\Delta t) = l \mid RC(t) = k) \quad (5)$$

and

$$PD_k(t) = P(default\ in\ [t; t+\Delta t] \mid RC(t) = k). \quad (6)$$

Consequently, one could therefore conclude that the unconditional PD term structure should be expressible in terms of the exponentiation of the one-period migration matrix $\bar{M}$ according to

$$\pi_k(t) = \frac{\sum_{l=1}^{N_r} \left[\bar{M}^{(t-t_0)/\Delta t}\right]_{k,l} \cdot \hat{\pi}_l}{\sum_{l=1}^{N_r} \left[\bar{M}^{(t-t_0)/\Delta t}\right]_{k,l}} \quad (7)$$

where $\hat{\pi}_k$ denotes here the masterscale PD assigned to rating class k. This method will be denoted as Exponentiation Approach in the following.

However, it turns out that the conceptual validity of this approach depends on the type of the underlying rating system. Basically, there are two very distinctive types of rating systems that shall briefly be discussed for the orientation of the reader. A PIT rating system intends to determine the obligor's PIT-PD being the probability of default in the considered time period given the macroeconomic situation. In contrast, a TTC rating system intends to determine the obligor's TTC-PD being the probability of default in the considered time period irrespective of the macroeconomic situation. While PIT-PD and TTC-PD are of similar magnitude in average economic scenarios, they may very strongly deviate from each other e.g. during an economic crisis, since the TTC-PD rating systems intends not to be influenced by such an overall economic downturn but to deliver stable TTC-PD estimates being constant over the economic cycle, while the PIT-PD intends to completely incorporate all economic indicators to deliver a most accurate prediction of the default probability at the considered point in time.

The crucial consideration with respect to the conceptual validity of the Exponentiation Approach is that migrations can in general be driven by two distinctive sources. They may be caused by individual changes in the credit risk of a given obligor. Such migrations will be referred to as idiosyncratic in the following. However, they may also be driven by systematic changes in the credit risk of all obligors due to a macroeconomic downturn, for instance. Such migrations will be referred to as systematic in the following. The key consideration is that a PIT rating system will always lead to migration matrices which incorporate the idiosyncratic as well as the systematic migrations. In contrast, there are no systematic migrations in a pure TTC rating system by definition.

As a consequence, the TTC-PD migration process can be considered as a homogeneous random walk process, meaning here in particular that the probabilities for an upward or downward rating migration of a given obligor do not depend on its preceding rating history. In this case the exponentiation of the one-period migration matrix, and thus the Exponentiation Approach, is a conceptually valid approach for computing the unconditional PD term structure as outlined above.

However, this statement does not hold for a PIT (or a hybrid PIT/TTC) rating system. A PIT-PD rating system fully reflects the macroeconomic situation. In particular, it will exhibit strong rating downgrades as a macroeconomic crisis evolves, rendering strong rating upgrades imminent as the crisis is finally resolved. The migration process observed in a PIT-PD rating system can therefore not be considered as a homogeneous random walk process in the sense that the probabilities for an upward or downward rating migration of a given obligor are not constant but do depend on its preceding rating history by virtue of the strong coupling to the macroeconomic situation. In this case the exponentiation of the one-period migration matrix, and thus the Exponentiation Approach, is a conceptually invalid approach for computing the PD term structure.

This argument is based on a qualitative mean-reversion picture of the economic cycle. However, the same argument also evolves from the Merton model as will be discussed by means of an illustrative example in the following. For this purpose the typical default condition $A(t) < \gamma(t)$ of the Merton model based on the asset value $A(t)$ is introduced here according to

$$\boldsymbol{A(t) = R \cdot X(t) + \sqrt{1 - R^2} \cdot \epsilon(t) < \gamma(t),} \tag{8}$$

where R^2 denotes the asset correlation, $X(t)$ is the systematic risk driver and $\varepsilon(t)$ is the obligor's idiosyncratic risk at time t − both being Standard-Gauss distributed and statistically independent on all discrete time points t (being separated by time span Δt) − and $\gamma(t)$ is the obligor's default threshold. An extended model allowing for a time-correlation of the systematic risk drivers X is discussed in the subsequent section. Following the discussion above there is a direct link between the latter threshold and the TTC-PD according to

$$\boldsymbol{PD_{TTC}(t) = \Phi(\gamma(t))}, \tag{9}$$

where $\boldsymbol{\Phi}$ denotes here the cumulative Standard-Gauss distribution.

For the sake of simplicity it will now be assumed for a moment that there are no idiosyncratic migrations meaning that the TTC-PD of all obligors does not change over time according to

$$\boldsymbol{\gamma(t) = \gamma(t_0)} \tag{10}$$

and that the macroeconomic state $\boldsymbol{X(t_0)}$ at time $\boldsymbol{t_0}$ is unknown (or alternatively, that the economic state is in equilibrium), meaning that PIT-PD and TTC-PD match each other at time $\boldsymbol{t_0}$. In such a scenario the PD term structure does not have a time dependence and can be written down trivially according to

$$\boldsymbol{\pi_k(t) = \pi_k(t_0) = \widehat{\pi}_k}. \tag{11}$$

In this example it shall further be assumed that the underlying rating system is a PIT-PD rating system. In the context of the introduced Merton model the PIT-PD at time t can be computed as

$$\boldsymbol{PD_{PIT}(t, X) = \Phi\left(\frac{\gamma - R \cdot X(t)}{\sqrt{1 - R^2}}\right)}. \tag{12}$$

In the assumed PIT-PD rating system the rating class reflects the PIT-PD and the observed rating migrations are directly driven by the change of the macroeconomic risk drivers X between time t and $t+\Delta t$. As a consequence there is a strong anti-correlation between migration events on consecutive time periods in the sense of a mean-reversion process as discussed above. The rationale is that the migrations are driven by a change in the PD and thus by $X_t - X_{t-1}$. The correlation of the migrations on consecutive time periods is thus directly related to $cor(X_t - X_{t-1}, X_{t+1} - X_t) \propto -var(X_t)$ being clearly negative since the macroeconomic risk drivers X are assumed here to be uncorrelated[13] Standard-Gauss variables.

Once the model is set up it can be evaluated numerically by means of a Monte-Carlo simulation program, which explicitly samples all random variables appearing in the asset value process in Eq. (8) for all obligors and for all considered time points $t = t_0, \dots, t_0 + N_t \cdot \Delta t$ with N_t=10 in this example. It then computes the PIT-PD according to Eq. (12) and assigns on the basis of the obligor's PIT-PD a rating class according to the credit institute's masterscale[14] for all considered time points and obligors. On the basis of the assigned rating classes the migration rate matrix $M^{obs}(t)$ is computed for all time periods leading finally to the migration matrix $\bar{M}$ as defined in Eq. (3). This Monte-Carlo iteration has been repeated 100 times for each set of model parameters discussed in the following leading to reasonably stable results for the presented migration matrices.

For the purpose of the numerical evaluation of the set up example a synthetic test portfolio consisting of 100,000 obligors has been composed with the performing obligors' TTC-PD distribution $\varrho_{TTC}(t_0)$ at time t_0 given in Figure 1a. Due to the assumption of time invariant TTC-PDs there are no migrations observable in the 'would-be' TTC migration matrix, which would result if a TTC-PD rating system had been used, presented in Figure 1b.

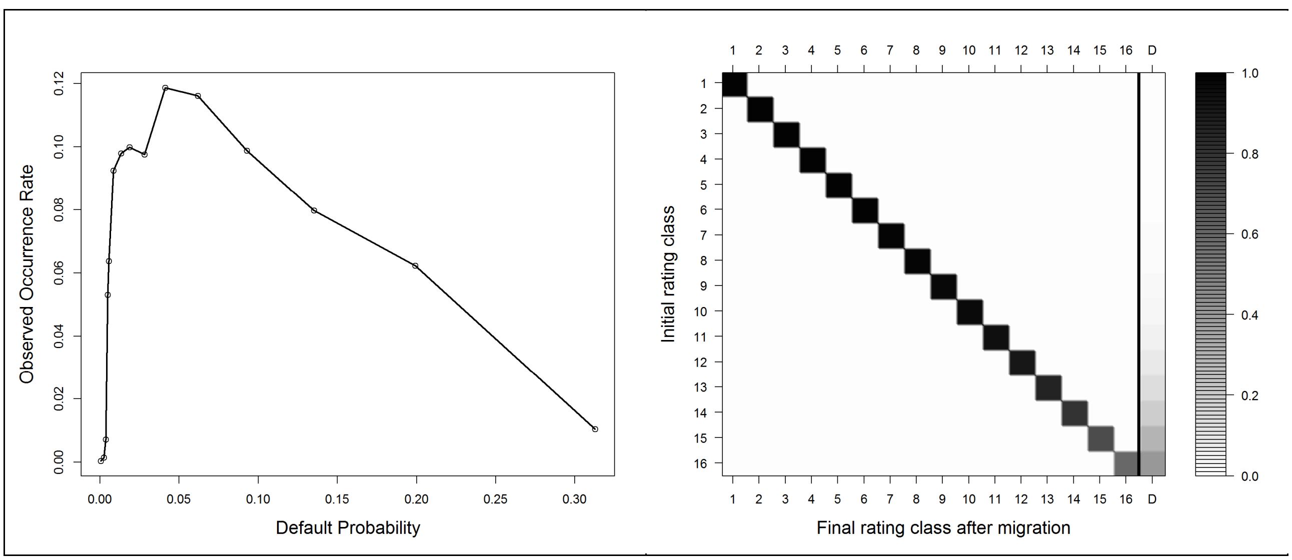


FIGURE 1: (A) TTC-PD DISTRIBUTION $\varrho_{TTC}(t_0)$ AT TIME t_0 OF THE SYNTHETIC TEST PORTFOLIO IS PRESENTED ON THE LEFT. (B) TTC-PD MIGRATION MATRIX INCLUDING THE DEFAULT VECTOR IS DISPLAYED ON THE RIGHT.

[13] In case of a temporal correlation of the systematic risk drivers, which will be introduced in the subsequent section, the aforementioned anti-correlation of migration events continues to exist but with respect to migration time horizons depending on the strength of the temporal correlation.

[14] In this example a hypothetical masterscale with 16 non-defaulted rating classes has been used.

The simulation program has been run with two different values for the asset correlation R^2. The results on the average PIT-PD migration matrix, i.e. $\overline{M}$, for the parameter setting R^2=1% are displayed in Figure 2a while the corresponding matrix for R^2=9% is presented in Figure 3a. The crucial observation is that even though there are no idiosyncratic migrations at all, which is manifest by the diagonal structure of the 'would-be' TTC-PD migration matrix displayed in Figure 1b, there is in fact a non-trivial migration structure observable in the PIT-PD migration matrices in Figure 2a and Figure 3a. Moreover, it can be inferred by comparing these results that the migration structure becomes more prominent with increasing values of the asset correlation R^2.

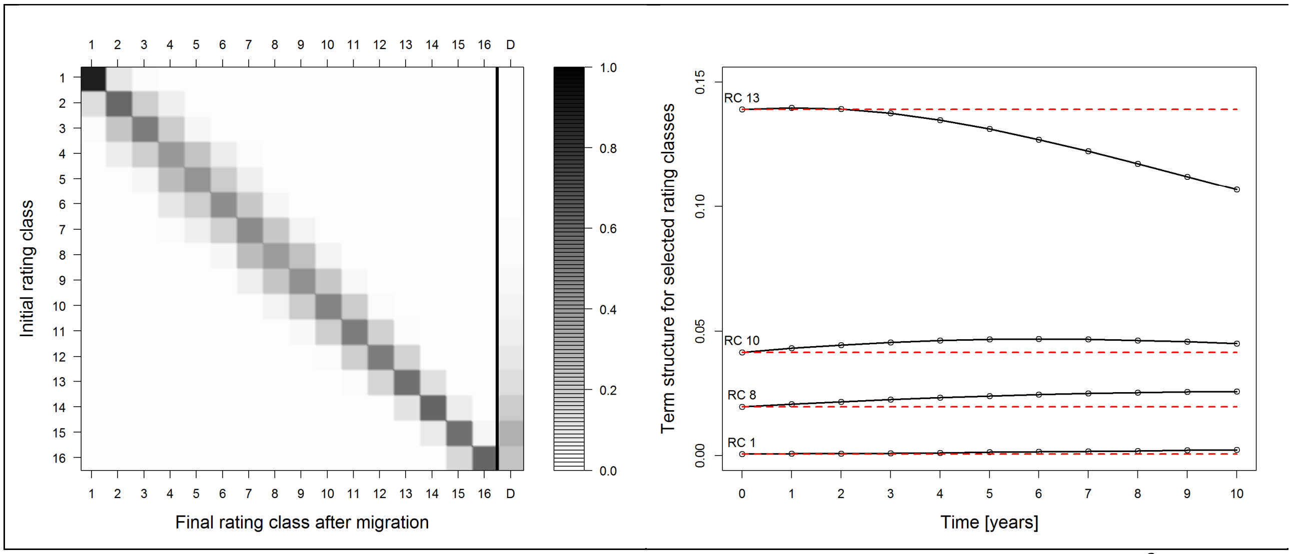


FIGURE 2: (A) PIT-PD MIGRATION MATRIX INCLUDING THE DEFAULT VECTOR IS DISPLAYED ON THE LEFT FOR PARAMETER R^2=1%. (B) THE CORRECT PD TERM STRUCTURE AS DEFINED IN EQ. (11) IS PRESENTED ON THE RIGHT BY THE RED DASHED LINES WHILE THE SOLID BLACK CURVES DISPLAY THE RESULT OF THE EXPONENTIATION APPROACH. BOTH TERM STRUCTURES ARE DISPLAYED FOR THE SELECTED RATING CLASSES 1, 8, 10, AND 13 OF THE UNDERLYING MASTERSCALE.

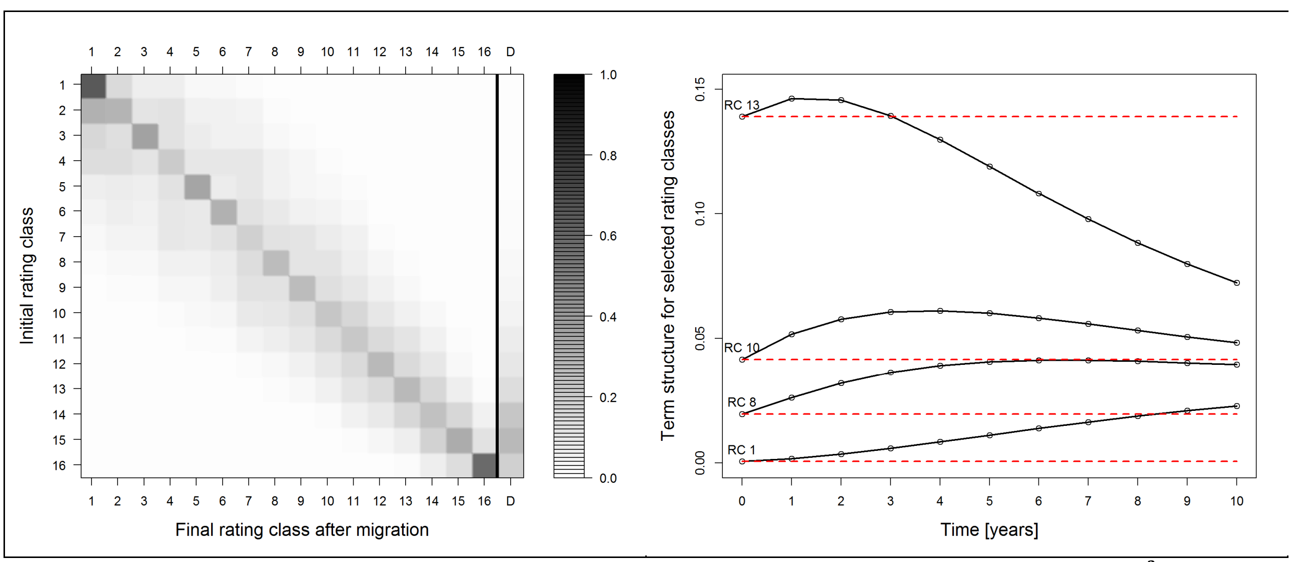


FIGURE 3: (A) PIT-PD MIGRATION MATRIX INCLUDING THE DEFAULT VECTOR IS DISPLAYED ON THE LEFT FOR PARAMETER R^2=9%. (B) THE CORRECT PD TERM STRUCTURE AS DEFINED IN EQ. (11) IS PRESENTED ON THE RIGHT BY THE RED DASHED LINES WHILE THE SOLID BLACK CURVES DISPLAY THE RESULT OF THE EXPONENTIATION APPROACH. BOTH TERM STRUCTURES ARE DISPLAYED FOR THE SELECTED RATING CLASSES 1, 8, 10, AND 13 OF THE UNDERLYING MASTERSCALE.

The relevance of this observation becomes evident when comparing the PD term structure that has been derived by applying the Exponentiation Approach as outlined in Eq. (7) with the correct result which is given in Eq. (11) for this example. For the parameter setting R^2=1% and R^2=9% the correct PD term structure and the Exponentiation Approach result are displayed for selected rating classes in Figure 2b and Figure 3b, respectively. From these presentations one can learn that the Exponentiation Approach to computing the PD term structure may – under certain circumstances – lead to significantly wrong results. In particular, one can observe that the Exponentiation Approach misleadingly suggests a very quick convergence of the rating classes with respect to their default probabilities especially for larger values of the asset correlation R^2. One may thus conclude that the average migration matrix should not be used to calculate the PD term structure by means of a simple matrix exponentiation unless the rating system is purely TTC and only the unconditional PD term structure is desired. In all other cases the Consistent Approach devised in the following should rather be used since it allows to consistently compute the unconditional as well as the conditional PD term structure independent of whether the rating system is PIT or TTC or a hybrid mixture thereof.

Consistent Approach to the Genuine Term Structure

For the orientation of the reader it shall be noted here that the example presented in the preceding section has to be considered somewhat simplistic to the extent that it has been assumed that

- there are no idiosyncratic migrations,
- there is no temporal correlation between the macroeconomic risk drivers X,
- the asset correlation R^2 is assumed to be the same for all obligors in the portfolio, and
- the underlying rating system is a pure PIT-PD rating system.

In practice any operational rating system will neither be a pure PIT-PD nor a pure TTC-PD rating system. In fact, it will always exhibit aspects of both rating system types. However, any operational rating system will always have a certain tendency towards one of these two extremes which can be thought of as a mixing ratio of PIT and TTC properties. Therefore, the Merton model discussed in the preceding section will be extended to address these issues.

In a first step, a new parameter is introduced here, which will be denoted as the 'PITness' parameter κ. It describes the type of a given rating system in terms of a mixing ratio of the PIT and TTC types. From now on, the default probability that a rating system assigns to a given obligor will be denoted as $PD_{RAT}(t,X)$ and it will be considered to be a linear combination of the obligor's PIT-PD and TTC-PD according to

$$\boldsymbol{PD_{RAT}(t,X) = \kappa \cdot PD_{PIT}(t,X) + (1-\kappa) \cdot PD_{TTC}(t)}. \qquad (13)$$

Analogously, the rating class $RC(t,X)$ assigned to an obligor will refer to $PD_{RAT}(t,X)$. In that sense the results in the preceding section have been derived for κ=1, i.e. for a pure PIT-PD rating system. In the following, however, results will be discussed for variable values of κ.

The criticism concerning the so far lacking temporal correlation of the systematic risk drivers is addressed by introducing the temporal correlation τ defined as

$$\boldsymbol{\tau = cor(X(t + \Delta t), X(t))} \tag{14}$$

meaning that the Gaussian random variables $X(t)$ specifying the macroeconomic situation at the considered time points $t = t_0, \dots, t_0 + N_t \cdot \Delta t$ will from now on be sampled according to a multi-dimensional Gauss distribution with correlation matrix given by the temporal correlation in Eq. (14).

The assumption of identical asset correlations R^2 for all obligors in the portfolio is addressed next. In practice one finds that a portfolio is typically rather inhomogeneous with respect to the industry, region, business size and many other aspects of the obligors. One would therefore expect the asset correlations R^2 of the obligors to be rather a distribution of values than a fixed value being the same for all obligors in the portfolio. To incorporate this view into the model the parameter $R = \sqrt{R^2}$ is considered a random variable for each obligor with probability density given by a beta-distribution with mean $\bar{R}$ and standard deviation σ. The beta-distribution has been chosen here for the sake of simplicity since it conveniently allows to parametrize probability distributions on the unit interval $[0; 1]$ with given average value and standard deviation.

Finally, the Merton model introduced in the preceding section is extended to incorporate also idiosyncratic migrations of the obligors, i.e. individual changes of the obligors' TTC-PD. This is done here by splitting up the aforementioned Merton approach into two process steps which are repeated sequentially as the evaluation of the model proceeds from time t to the consecutive time point $t + \Delta t$.

Firstly, the TTC-PD distribution $\varrho_{TTC}(t)$ of the performing obligors at time t is transformed into the TTC-PD distribution $\varrho'_{TTC}(t + \Delta t)$ at time $t + \Delta t$ according to

$$\boldsymbol{\varrho'_{TTC}(t + \Delta t) = \Upsilon(\varrho_{TTC}(t), \Sigma)} \tag{15}$$

where the mapping Υ simply applies the idiosyncratic migration probabilities, i.e. excluding any default probabilities, given by the TTC-PD rating class migration matrix Σ on the TTC-PD distribution $\varrho_{TTC}(t)$. For clarification it is remarked that the latter step involves mapping the obligors' TTC-PDs to rating classes according to the masterscale, applying the matrix Σ to this discretization of the TTC-PD distribution, and using the masterscale again for back-transforming the migrated rating classes to TTC-PDs thus yielding the (discretized) TTC-PD distribution $\varrho'_{TTC}(t + \Delta t)$.

Having in mind a parametrization of the so far unspecified TTC-PD migration matrix Σ in a compact but sufficiently flexible manner, two parameters λ and ν are introduced here according to

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma_{1,1}} & \cdots & \boldsymbol{\sigma_{1,N_r}} \\ \vdots & \ddots & \vdots \\ \boldsymbol{\sigma_{N_r,1}} & \cdots & \boldsymbol{\sigma_{N_r,N_r}} \end{pmatrix} \tag{16}$$

with

$$\boldsymbol{\sigma_{k,l}} = \frac{\boldsymbol{\lambda^{(|k-l|^{\nu})}}}{\sum_{j=1}^{N_r} \boldsymbol{\lambda^{(|k-j|^{\nu})}}} \, . \tag{17}$$

Secondly, the asset value at time $t + \Delta t$ is sampled for each obligor and the default condition of the obligors is determined according to Eq. (8), however, on the basis of the default threshold $\gamma(t + \Delta t)$ corresponding to the migrated TTC-PD distribution $\varrho'_{TTC}(t + \Delta t)$. Finally, the TTC-PD distribution $\varrho_{TTC}(t + \Delta t)$ of the performing obligors at time $t + \Delta t$ is determined by removing the defaulted obligors from $\varrho'_{TTC}(t + \Delta t)$.

Once this extended multi-period Merton model including the idiosyncratic migrations is set up and its free parameters $\kappa, \lambda, \nu, \bar{R}, \sigma,$ and τ are fixed this model can be evaluated numerically, allowing to compute the unconditional term structure $\pi_k(t)$. By virtue of the incorporated temporal correlation τ and the PITness parameter κ the newly introduced model also allows to compute the conditional term structure $\pi_k(t, x_0)$ given the macroeconomic situation x_0 in the sense of a concrete realization of the macroeconomic random variable $X(t_0)$ at time t_0 as defined in Eq. (2) according to

$$\boldsymbol{\pi_k(t, x_0) = E(PD_{PIT}(t, X) \mid RC(t_0, X) = k \wedge alive\ at\ t \wedge X(t_0) = x_0)} \quad (18)$$

with $E(A|B)$ denoting here the expectation value of the random variable A given the event B.

For the orientation of the reader the economic relevance of the latter quantity, which will be referred to as genuine term structure in the following, shall briefly be addressed. Assuming that the current macroeconomic situation represented by $X(t_0) = x_0$ corresponds to a recession period one can expect that, at the moment, the PIT-PDs of the obligors, and thus the observed default rates, are much higher than their respective TTC-PDs. However, looking ahead a few years one would expect that the crisis will finally be resolved and that the obligors' PIT-PDs, and thus the observed default rates, will eventually decrease as they move towards their TTC-PDs. Assuming, however, a currently booming macroeconomic situation one would expect the default rates to increase in a few years according to the same rationale. The reason is that the expectation value of the future macroeconomic risk driver given the current macroeconomic situation $X(t_0) = x_0$, i.e. $E(X(t) \mid X(t_0) = x_0)$, is assumed to relax from its current value x_0 to an average scenario, i.e. towards zero, with the relaxation speed controlled by the temporal correlation parameter τ.

The genuine term structure therefore actually depends on the starting condition, i.e. the current macroeconomic situation $X(t_0) = x_0$, with a strong dependence on the starting condition in the short-term region, while the genuine term structure $\pi_k(t, x_0)$ converges approximately to the unconditional term structure $\pi_k(t)$ in the long term.

Fixation of the free Parameters

The genuine term structure defined in Eq. (18) can be evaluated numerically following the process steps discussed in the preceding section on the basis of the free parameters $\kappa, \lambda, \nu, \bar{R}, \sigma,$ and τ which, however, are generally unknown. Prior to any attempt of computing the genuine term structure all of these free parameters thus have to be fixed. For this purpose a fitting procedure allowing to determine the aforementioned parameters by analyzing historic migration data has been devised. This approach will be briefly sketched in the following.

The main idea here is to use a maximum likelihood approach on the basis of historically observed migration event count matrices $N(t)$ defined as

$$\boldsymbol{N(t)} = \begin{pmatrix} \boldsymbol{n_{1,1}(t)} & \dots & \boldsymbol{n_{1,N_r}(t)} & \boldsymbol{d_1(t)} \\ \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{n_{N_r,1}(t)} & \dots & \boldsymbol{n_{N_r,N_r}(t)} & \boldsymbol{d_{N_r}(t)} \\ \boldsymbol{0} & \dots & \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} \tag{19}$$

where $n_{k,l}(t)$ denotes the number of obligors that have migrated historically from rating class k to rating class l in the time period $[t, t+\Delta t]$ and $d_k(t)$ is the number of defaults in that time period. The likelihood function $L(\kappa, \lambda, \nu, \bar{R}, \sigma, x, \tau, N)$ that is supposed to be maximized with respect to the free parameters κ, λ, ν, $\bar{R}$, σ, x, and τ is given as

$$\boldsymbol{log\left(L(\kappa,\lambda,\nu,\bar{R},\sigma,x,\tau,N)\right) = Const + \ log\left(\delta(x,\tau)\right) + \sum_{k,l=1}^{N_r}\sum_t n_{k,l}(t)\cdot log\left(m_{k,l}^{theo}(t)\right) + \sum_{k=1}^{N_r}\sum_t d_k(t)\cdot log\left(m_{k,N_r+1}^{theo}(t)\right)} \tag{20}$$

where the vector x denotes a considered realization of the systematic random variables X and the constant $Const$ is ignored in the following. The theoretical migration and default probabilities $m_{k,l}^{theo}$ are computed on the basis of the extended multi-period Merton model discussed in the previous section given a TTC-PD distribution $\varrho_{TTC}(t) = \bar{\rho}_{TTC}$ which is assumed to be the same on all time periods due to a continuous new-deal assumption. The so far unspecified contribution $\delta(x,\tau)$ contains here the multidimensional Gauss density of the systematic variables x for the temporal correlation τ as well as an integration volume contribution.

When analyzing a given migration history $N(t)$ the latter TTC-PD distribution $\bar{\rho}_{TTC}$ is constructed in a first step before the actual likelihood optimization. This is done by forming homogeneous obligor groups with respect to their rating PD in each time period and tracking these groups over the considered time periods by means of a quantile mapping with respect to their rating PD, e.g. for each time period the best 10% of the obligors are grouped together and identified as the same obligor group. This allows to identify an equivalent obligor group in each considered time period irrespective of the fluctuation of the macroeconomic variables X and thus of the absolute level of the rating PD. For these homogeneous obligor groups the observed default rate is averaged over the considered time periods and the resulting average PD is assigned to the respective obligor group as the group's TTC-PD, which directly yields the TTC-PD distribution $\bar{\rho}_{TTC}$.

It is noteworthy that the fitting approach aims at fixing all free parameters $\kappa, \lambda, \nu, \bar{R}, \sigma, x,$ and τ including in particular the vector of the systematic risk drivers x on all considered time periods. However, performing the optimization of all parameters in a single step results in rather unstable results. Significant effort thus has to be invested in finding a stable fitting scheme. For instance, it turns out that the systematic risk drivers x and the temporal correlation τ can be determined in a separate fit that considers default information only. In a second step, the remaining parameters $\kappa, \lambda, \nu, \bar{R}, \sigma$ are then fitted on the basis of the full migration and default information.

The quality of our implemented fit procedure shall be discussed in the following. To assess the achieved fit accuracy a set of synthetic migration event count time series $N^{synth}(t)$ has been constructed according to the extended multi-period Merton model introduced in the previous section with a given average TTC-PD distribution $\bar{\rho}_{TTC}$, i.e. $\varrho_{TTC}(t) = \bar{\rho}_{TTC}$ at all considered time points. This has been done for various parameter settings of $\kappa, \lambda, \nu, \bar{R}, \sigma, \tau$, and randomly sampled macroeconomic risk drivers x which will be referred to as *true parameters* in the following. The synthetically sampled migration matrices $N^{synth}(t)$ are then analyzed by means of the implemented fit procedure and the obtained fit results for the parameters of $\kappa, \lambda, \nu, \bar{R}, \sigma, x$, and τ are referred to as *observed parameters*. The quality of the fit procedure is then assessed by comparing the *true parameters* with the *observed parameters*.

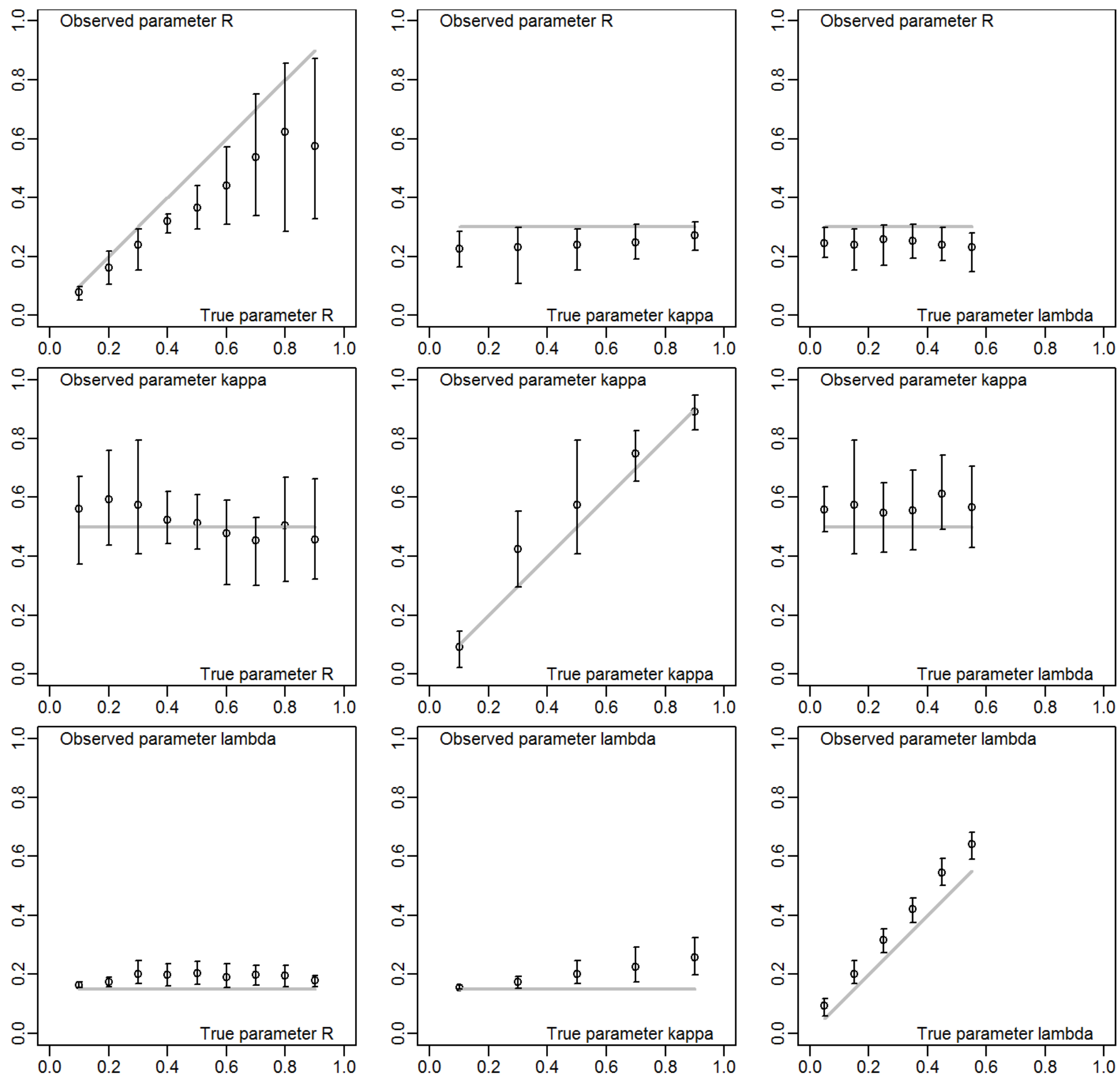


FIGURE 4: IN THE FIRST COLUMN THE PARAMETERS $\kappa = 0.50$ AND $\lambda = 0.15$ ARE FIXED WHILE $\bar{R}$ IS VARIED. IN THE SECOND COLUMN THE PARAMETERS $\bar{R} = 0.30$ AND $\lambda = 0.15$ ARE FIXED WHILE κ IS VARIED. IN THE THIRD COLUMN THE PARAMETERS $\bar{R} = 0.30$ AND $\kappa = 0.50$ ARE FIXED WHILE λ IS VARIED. FOR EACH DATA POINT 60 INDEPENDENT ITERATIONS OF SAMPLING AND MEASUREMENT HAVE BEEN PERFORMED. THE PLOTTED DATA POINTS SHOW THE AVERAGE VALUE OF THE MEASURED PARAMETER. THE ERROR BARS MARK THE 15%- AND 85%-QUANTILES OF THE SINGLE MEASUREMENTS. THE SOLID LINE SHOWS THE TRUE PARAMETER.

The synthetic migration histories have been sampled for a fictitious portfolio with $N_{obs} = 100{,}000$ obligors, $N_t = 10$ periods, no temporal correlation, i.e. $\tau = 0$, $\upsilon = 0.60$, a smeared distribution of the asset correlations with $\sigma = 0.15$, and a given average TTC-PD distribution $\bar{\rho}_{TTC}$ presented in Figure 1a. For the remaining parameters κ, λ, $\bar{R}$ various settings have been tested. For each parameter setting 60 independent iterations of sampling the migration history according to the true parameters and measuring the model parameters have been performed. The comparison between the true and the observed parameters κ, λ, $\bar{R}$ is presented in Figure 4, while the average observed values for the fixed parameters τ, υ, σ over all simulation iterations are $\tau = 0.071 \pm 0.131$, $\upsilon = 0.692 \pm 0.092$, and $\sigma = 0.158 \pm 0.098$ where the uncertainty given here is the standard deviation of the single measurement.

From the results presented in Figure 4 one can infer that the fitting process is a suitable yet slightly biased estimator for the true parameters underlying the sampling of the synthetic migration data; the magnitude of the observable bias − especially for the parameters λ and $\bar{R}$ − can be considered comparable to the statistic uncertainty of the single measurement. Especially for the measurement of the asset correlation it is well-known that current standard approaches based on the analysis of historic default rate time series are typically biased estimators[15,16]. While the removal of the bias from the parameter estimation scheme would certainly add value to the overall methodology, the measurement accuracy achieved so far is considered satisfactory for the first explorative analysis conducted here.

A Real World Example

The introduced methodology for constructing the genuine term structure shall now be applied to real world migration data. For that purpose the GCD consortium provided anonymized historic migration and default time series. In the sense of a first explorative study only the facility asset class 'Large Corporates' as a sub-portfolio of the rich GCD data has been analyzed.

The GCD migration and default data are available on different granularity levels, i.e. on an annual, a quarterly, as well as a multi-year time horizon. Here the annual data with non-overlapping time periods starting at the end of December of a given year, for instance starting in December 2008 noted as '2008-12', has been selected.

A major reason for selecting the facility class 'Large Corporates' is the high data availability of the asset class on a global scale. In Figure 5a the total obligor count in that sub-portfolio is plotted versus the respective time period. One can observe that this selected sub-portfolio contains more than 50,000 obligors in the recent years and that the total obligor count has been growing quickly in the past. The plot also reveals that the data before the period '2007-12' have a significantly lower statistic foundation than the periods thereafter and should thus be considered more carefully.

[15] M. Gordy, E. Heitfield: Estimating factor loadings when ratings performance data are scarce. Memorandum, Board of Governors of the Federal Reserve System, 2000.
[16] K. Duellmann, J. Küll, M. Kunisch: Estimating asset correlations from stock prices or default rates - which method is superior?. Journal of Economic Dynamics & Control, Elsevier, 34 (11), 2010

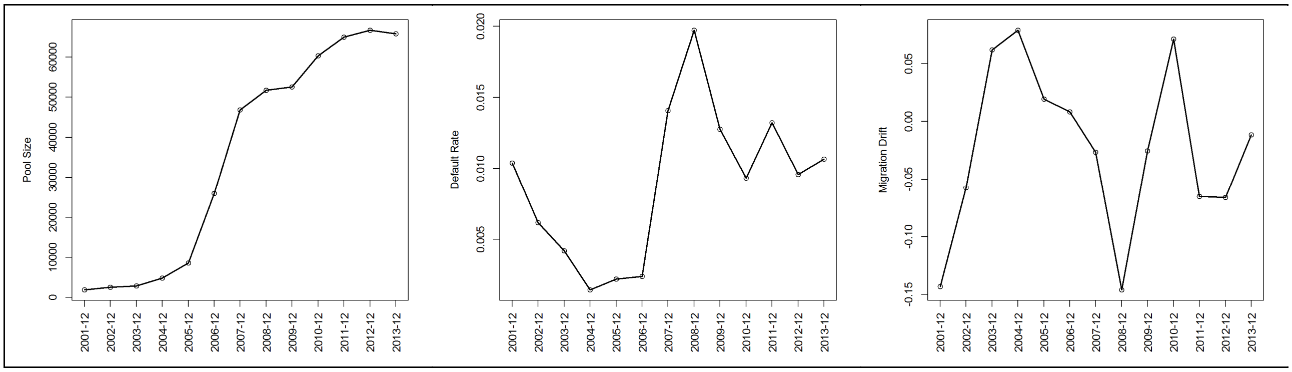
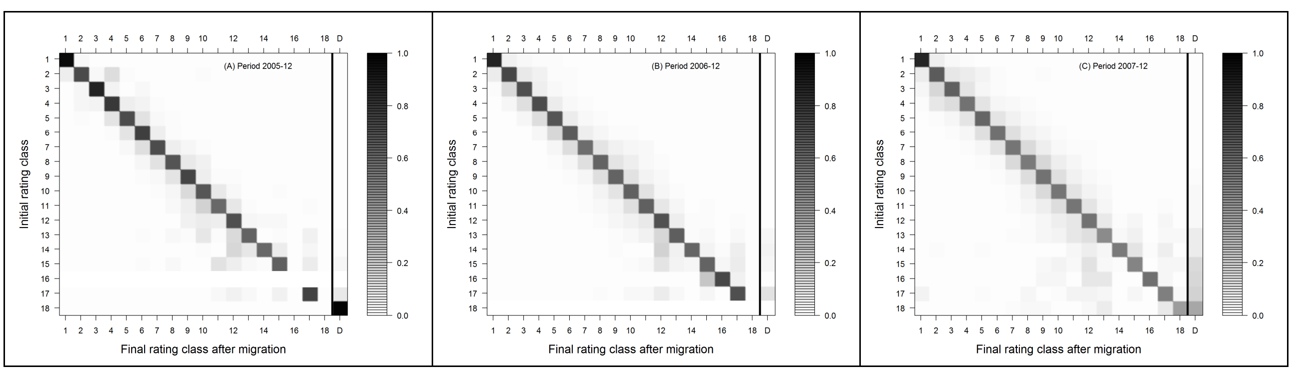


FIGURE 5: (A) THE TOTAL OBLIGOR COUNT OF THE CONSIDERED SUB-PORTFOLIO WITH ASSET FACILITY CLASS 'LARGE CORPORATES' IS SHOWN ON THE LEFT FOR THE CONSIDERED TIME PERIODS. (B) THE CORRESPONDING HISTORIC DEFAULT RATE TIME SERIES IS PLOTTED IN THE MIDDLE. (C) THE MIGRATION DRIFT AS DEFINED IN EQ. (21) IS SHOWN ON THE RIGHT.

The default rates observed in this sub-portfolio over the considered time periods are presented in Figure 5b. As expected the plot shows a major peak in the period '2008-12' corresponding to the world financial crisis. Closely related to the actual default behavior of the obligors in the portfolio is the evolution of the up- and downgrades of the corresponding credit ratings. To monitor the latter rating grade evolution the so-called migration drift $\mu(t)$ is introduced here according to

$$\boldsymbol{\mu(t)} = \frac{\sum_{k,l,k>l}^{N_r} n_{k,l}(t) - \sum_{k,l,k<l}^{N_r} n_{k,l}(t)}{\sum_{k,l}^{N_r} n_{k,l}(t)}. \qquad (21)$$

The migration drift thus specifies the difference between the rate of upgrades and downgrades in a given period, such that a positive value indicates an improvement of the obligors' rating grades. The migration drift measured on the considered GCD data is presented in Figure 5c. As expected one observes a sharp negative peak in the period '2008-12' corresponding to the world financial crisis. In general, one can infer from comparing the presented default rate and migration drift time series that the historic rating grade evolution mimics well the observed default behavior of the obligors.



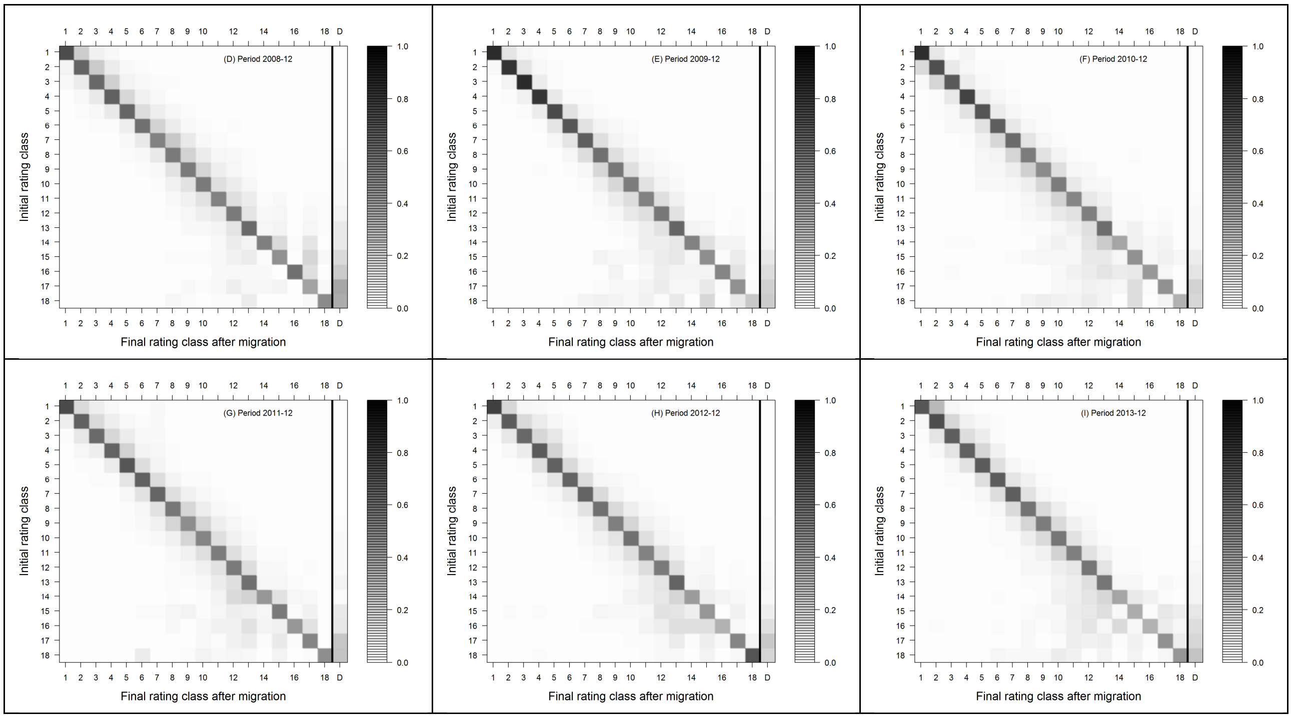


FIGURE 6: (A),…,(I) THE MIGRATION RATE MATRICES OBSERVED IN THE PERIODS '2005-12' TO '2013-12' ARE PRESENTED.

For the orientation of the reader the migration rate matrices of the most recent nine years, i.e. from period '2005-12' to '2013-12', are shown in Figure 6. For the first two presented periods, i.e. '2005-12' and '2006-12', one can observe some missing data effects in the representation due to empty matrix rows. However, as the total obligor count steadily increases over time such data availability issues are resolved in the following years. It should also be mentioned here that the first three rating classes, i.e. the rating classes with the best credit-worthiness, have been removed from the analysis due to low occupation numbers. Apart from these remarks one can learn from the presentation in Figure 6 how the annual migration drifts displayed in Figure 5c are related to the corresponding migration matrices.

In principle, the established methodology for deriving the genuine term structure can now directly be applied to the available migration history. However, in contrast to the synthetic data sampled in the previous sections the real world portfolio consists of obligors from different industries and countries, thus naturally coupling to different systematic risk drivers that are to a significant extent independent of each other especially in the absence of an economic crisis. In such a setting the systematically driven migrations on the aforementioned sub-segment level, e.g. industry and country level, misleadingly appear to be idiosyncratic in the overall portfolio view, since the individual systematic effects on the respective sub-segment point into different directions. Therefore, the best way for clearly separating the systematic from the idiosyncratic migrations is to study phases of global economic stress, since in such scenarios the systematic drivers in nearly all sub-segments point into the same direction, thus acting as if there were only one systematic driver. Migrations against that global trend are then clearly idiosyncratically driven.

Following the aforementioned rationale the historic realization of the systematic risk factors x and the corresponding temporal correlation τ is determined by fitting the default rate time series over all available time periods as discussed in section 'Fixation of the free Parameters', while the remaining parameters κ, λ, ν, $\bar{R}$, σ are fixed by fitting the one time period with the most economic stress only, i.e. the period ‘2008-12’.

κ	λ	ν	$\bar{R}$	σ	τ
0.448	0.160	0.571	0.227	0.368	0.669

TABLE 1: FIT RESULTS FOR THE REMAINING PARAMETERS κ, λ, ν, $\bar{R}$, σ, AND τ FROM ANALYZING THE CRISIS TIME PERIOD ‘2008-12’ AS EXPLAINED IN THE MAIN TEXT.

The results of the fit procedure for the parameters κ, λ, ν, $\bar{R}$, σ, and τ are presented in Table 1 and the fixed realization of the historic risk drivers x is plotted in Figure 7a. As expected the crisis period ‘2008-12’ as well as the economic boom years before the crisis can clearly be identified.

Once the model parameters have been fixed, the genuine term structure depending on a given economic start condition can directly be computed. This is done here for the exemplarily chosen start conditions of the periods ‘2006-12’ and ‘2008-12’. These two periods have been chosen since they represent an economic boom phase and an economic crisis as start condition, respectively. The resulting genuine term structures for these selected start conditions are presented in Figure 7b,c by the red dashed curves. These plots moreover compare the latter results with the corresponding estimates from the Exponentiation Approach depicted here by the black solid lines.

The term structure derived from the Exponentiation Approach does not depend on the economic starting condition. This is reflected in the black solid lines showing exactly the same behavior for each rating class in Figure 7b and in Figure 7c. The genuine term structure derived from the newly introduced methodology, however, depends crucially on the start condition. In Figure 7b the start condition is an economic boom year. Therefore, the PD predicted by the genuine term structure rises for the subsequent years since the economic boom is expected to relax to an average economic situation. Contrarily, the start condition is an economic crisis in Figure 7c. Therefore, the PD predicted by the genuine term structure decreases or rises less quickly than the Exponentiation Approach would suggest, since the economic crisis is expected to be mitigated in the following years.

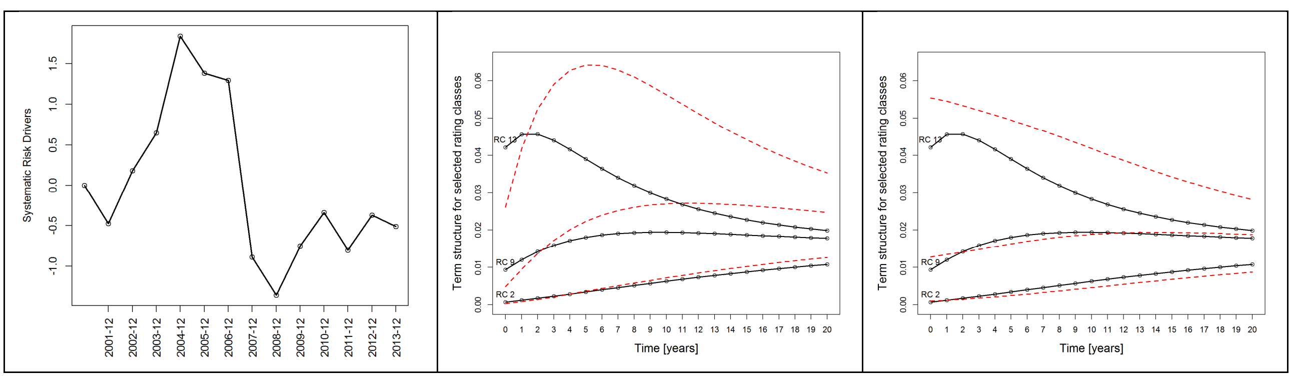


FIGURE 7: (A) THE FIT RESULT OF THE HISTORIC REALIZATION OF THE SYSTEMATIC RISK DRIVER x IS PRESENTED ON THE LEFT. (B) THE GENUINE TERM STRUCTURE (RED DASHED CURVE) AND THE CORRESPONDING RESULT FROM THE EXPONENTIATION APPROACH (BLACK SOLID LINE) FOR THE SELECTED RATING CLASSES 2, 9, AND 13 ARE PLOTTED FOR THE START CONDITION OF THE ECONOMIC BOOM PERIOD ‘2006-12’ IN THE MIDDLE. (C) THE SAME CURVES AS IN (B) ARE PRESENTED ON THE RIGHT BUT FOR THE START CONDITION OF THE ECONOMIC CRISIS PERIOD ‘2008-12’.

The observed discrepancy between both approaches in the very first period is caused by the determined classification of the rating system. According to the measured value for the PITness parameter κ the rating system is assumed to deliver neither the PIT-PD nor the TTC-PD but a value roughly halfway in between. While the Exponentiation Approach reproduces the rating PD in the very first time period, the genuine term structure aims at delivering the actual default probability, i.e. the pure PIT-PD.

Moreover, it should be noted that the Exponentiation Approach as well as the new methodology both result in term structure curves that converge over time. However, the convergence velocity is significantly higher for the Exponentiation Approach since it misleadingly incorporates systematic migrations into the exponentiation matrix.

The impact of the newly introduced methodology on the term structure becomes the more prominent the closer the underlying rating system comes to a perfect PIT-PD rating system, i.e. the closer the PITness parameter κ approaches 1. To demonstrate this relationship the previous analysis is repeated with migration data that exhibit a more PIT-like behavior.

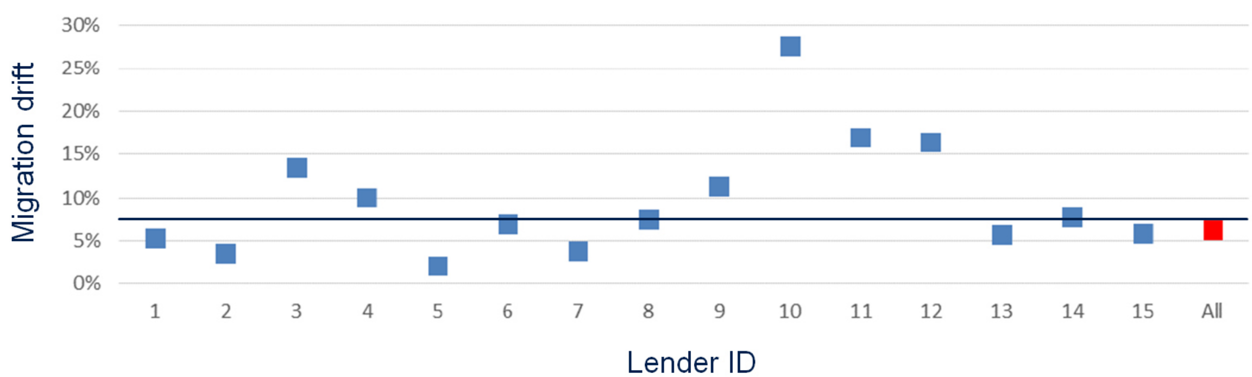


FIGURE 8: THE TIME-AVERAGED ABSOLUTE MIGRATION DRIFT $\langle|\mu|\rangle_t$ PER CREDIT INSTITUTE CONTRIBUTING TO THE GCD DATA POOL IS PRESENTED IN ANONYMIZED FORM. THE INSTITUTES ARE REPRESENTED HERE BY A LENDER ID. THE SOLID HORIZONTAL LINE INDICATES THE THRESHOLD FOR SELECTING THE DATA FOR THE MORE PIT-LIKE MIGRATION HISTORY. THE RED BOX ON THE VERY RIGHT INDICATES THE PORTFOLIO AVERAGE OF THE ABSOLUTE MIGRATION DRIFT.

For this purpose the available GCD data has been segmented on the basis of the time-averaged absolute migration drift $\langle|\mu|\rangle_t$ per credit institute contributing to the GCD data pool. In Figure 8 the latter quantity is presented for the 15 contributing institutes in anonymized form. Those credit institutes exhibiting larger migration drifts are likely to operate rating systems with higher degree of PITness κ. Here the new migration history is constructed by constraining the overall data set to the sub-portfolio spanned by those credit institutes with an average migration drift above 7.5%, i.e. $\langle|\mu|\rangle_t > 7.5\%$.

The total obligor count, the default rate, and the migration drift history for the new migration time series obtained by this segmentation approach are presented in Figure 9.

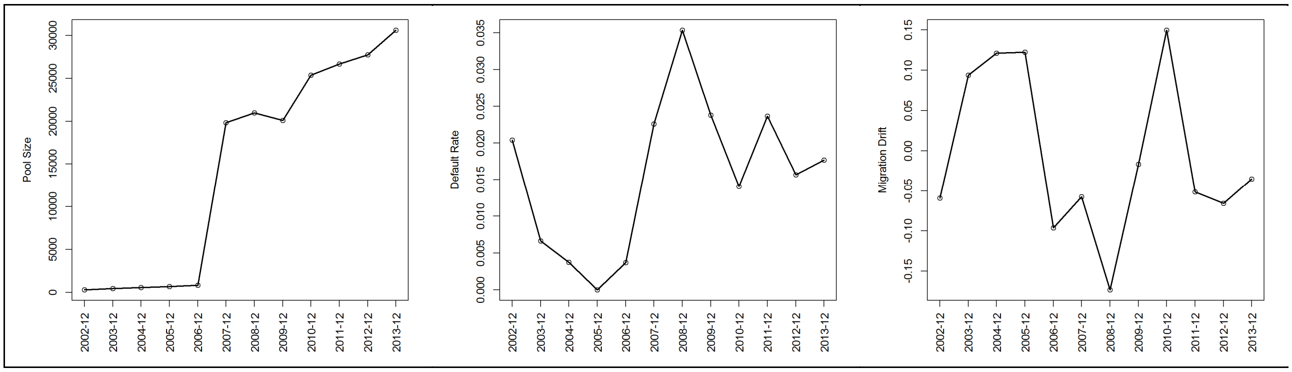


FIGURE 9: (A) THE TOTAL OBLIGOR COUNT OF THE CONSIDERED MORE PIT-LIKE MIGRATION HISTORY IS SHOWN ON THE LEFT FOR THE CONSIDERED TIME PERIODS. (B) THE CORRESPONDING HISTORIC DEFAULT RATE TIME SERIES IS PLOTTED IN THE MIDDLE. (C) THE MIGRATION DRIFT AS DEFINED IN EQ. (21) IS SHOWN ON THE RIGHT.

Applying the identical fitting strategy as before, i.e. fixing the systematic risk drivers x and τ on the basis of the default rate time series and the remaining parameters κ, λ, ν, $\bar{R}$, σ by fitting the crisis period '2008-12', the model parameters have been determined and are presented in Table 2 and Figure 10a, respectively.

κ	λ	ν	$\bar{R}$	σ	τ
0.775	0.224	0.734	0.364	0.424	0.510

TABLE 2: FIT RESULTS FOR THE REMAINING PARAMETERS κ, λ, ν, $\bar{R}$, σ, AND τ FROM ANALYZING THE CRISIS TIME PERIOD '2008-12' OF THE SEGMENTED MIGRATION DATA SET AS EXPLAINED IN THE MAIN TEXT.

As expected the PITness parameter κ is significantly larger than in the previous analysis indicating that the segmented migration data exhibits indeed a more PIT-like behavior as intended. The corresponding genuine term structures for selected rating classes given the start conditions of the periods '2006-12' and '2008-12' are plotted in Figure 10b,c.

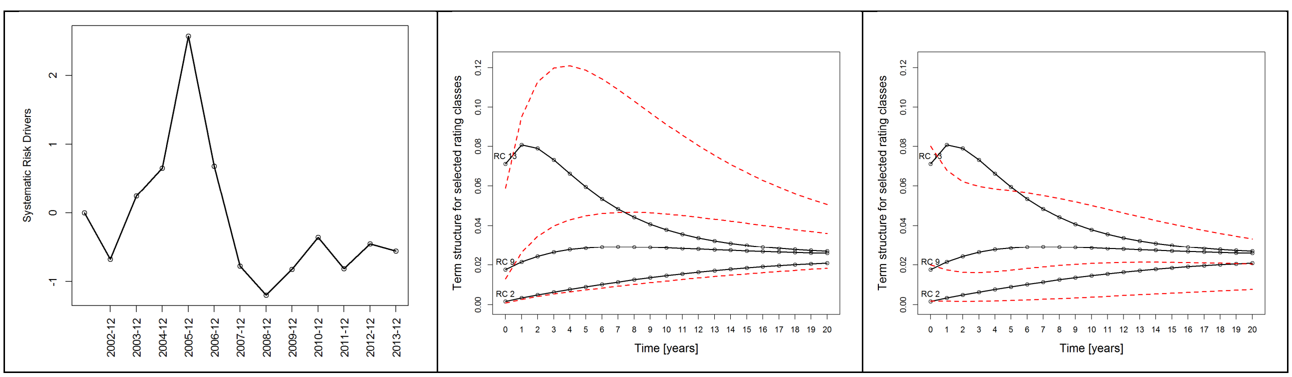


FIGURE 10: (A) THE FIT RESULT OF THE HISTORIC REALIZATION OF THE SYSTEMATIC RISK DRIVER x IS PRESENTED ON THE LEFT FOR THE CONSIDERED MORE PIT-LIKE MIGRATION HISTORY. (B) THE GENUINE TERM STRUCTURE (RED DASHED CURVE) AND THE CORRESPONDING RESULT FROM THE EXPONENTIATION APPROACH (BLACK SOLID LINE) FOR THE SELECTED RATING CLASSES 2, 9, AND 13 ARE PLOTTED FOR THE START CONDITION OF THE ECONOMIC BOOM PERIOD '2006-12' IN THE MIDDLE. (C) THE SAME CURVES AS IN (B) ARE PRESENTED ON THE RIGHT BUT FOR THE START CONDITION OF THE ECONOMIC CRISIS PERIOD '2008-12'.

In comparison to the previous results the impact of the systematic risk drivers on the term structure has become more prominent according to the larger PITness degree κ. This is particularly well observable in Figure 10c where the assumed relaxation from the crisis drives the term structure towards lower default probabilities while the Exponentiation Approach predicts rising PDs for the first years.

Conclusions

From the given example calculations one can learn that the widely spread Exponentiation Approach can lead to significantly different results with respect to the term structure than the newly introduced construction scheme. These differences are of quantitative as well as of qualitative nature, since the new method incorporates in a conceptually consistent manner the dependence on the economic start condition − in contrast to the Exponentiation Approach. However, more research is necessary to further investigate the model implications and its optimal parametrization and calibration.